THE BORDER RANK OF THE MULTIPLICATION OF 2×2 MATRICES IS SEVEN

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1. Introduction

One of the leading problems of algebraic complexity theory is matrix multiplication. The naïve multiplication of two $n \times n$ matrices uses n^3 multiplications. In 1969 Strassen [20] presented an explicit algorithm for multiplying 2×2 matrices using seven multiplications. In the opposite direction, Hopcroft and Kerr [12] and Winograd [22] proved independently that there is no algorithm for multiplying 2×2 matrices using only 6 multiplications.

The precise number of multiplications needed to execute matrix multiplication (or any given bilinear map) is called the *rank* of the bilinear map. A related problem is to determine the *border rank* of matrix multiplication (or any given bilinear map), first introduced in [6, 5]. Roughly speaking, some bilinear maps may be approximated with arbitrary precision by less complicated bilinear maps and the border rank of a bilinear map is the complexity of arbitrarily small "good" perturbations of the map. These perturbed maps can give rise to fast exact algorithms for matrix multiplication, see [7]. The border rank made appearances in the literature in the 1980's and early 90's, see, e.g., [6, 5, 19, 8, 15, 2, 10, 11, 4, 3, 17, 16, 18, 1, 9, 21], but to our knowledge there has not been much progress on the question since then.

More precisely, for any complex projective variety $X \subset \mathbb{CP}^N = \mathbb{P}V$, and point $p \in \mathbb{P}V$, define the *X-rank of p* to be the smallest number r such that p is in the linear span of r points of X. Define $\sigma_r(X)$, the *r-th secant variety of X*, to be the Zariski closure of the set of points of X-rank r, and define the *X-border rank of p* to be the smallest r such that $p \in \sigma_r(X)$. The terminology is motivated by the case $X = Seg(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}) \subset \mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b)$, the *Segre variety* of rank one matrices. Then the X-rank of a matrix is just its usual rank.

Let A^*, B^*, C be vector spaces and let $f : A^* \times B^* \to C$ be a bilinear map, i.e., an element of $A \otimes B \otimes C$. Let $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A \otimes B \otimes C)$ denote the *Segre variety* of decomposable tensors in $A \otimes B \otimes C$. The border rank of a bilinear map is its X-border rank. While for the Segre product of two projective spaces, border rank coincides with rank, here they can be quite different.

In this paper we prove the theorem stated in the title. Let $MMult \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ denote the matrix multiplication operator for two by two matrices. Strassen [18] showed that $MMult \notin \sigma_5(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$. Our method of proof is to decompose $\sigma_6(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3) \backslash \sigma_5(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ into sixteen components, and then using case by case arguments show $MMult$ is not in any of the components. The decomposition rests upon a differential-geometric understanding of curves in submanifolds, which is carried out in §2. In §3 we roughly describe the components of $\sigma_r(X)$ for an arbitrary variety, and give a precise description for $\sigma_6(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, giving a normal form for a point of each component. In §4 we carry out our case by case analysis.

§5 consists of a treatment of the cases that were overlooked in an earlier version of this article.

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2. Taylor series for curves on submanifolds

Let V be a vector space and $\mathbb{P}V$ the associated projective space of lines through the origin in V. If $x \in V$ we let $\hat{x} \subset V$ denote the line through x and $[x] \in \mathbb{P}V$ the corresponding point in projective space. If $Z \subset \mathbb{P}V$ is a set, we let $\hat{Z} \subset V$ denote the corresponding cone in V.

We begin with some very general local differential geometry:

Lemma 2.1. *Let $X \subset \mathbb{P}V$ be an analytic submanifold. Let $[x_0] \in X$ and choose a splitting $V = \hat{x}_0 \oplus T_{[x_0]}X \oplus N_{[x_0]}X$. (This is necessary because in affine and projective geometry the tangent and normal spaces are only well defined as quotient spaces of V.) Having done so, we may and will identify abstract and embedded tangent spaces.*

Let $x(t) \subset V$ be an analytic curve on $\hat{X}$ such that $x(0) = x_0$. Let $F_j \in S^j T^*_{[x_0]}X \otimes N_{[x_0]}X$ denote the j-th Fubini form of X at $[x_0]$ (see [13] p. 107 for a definition or one can use the coordinate definition given in the proof below). Write, in local coordinates

$$x(t) = x_0 + tx_1 + t^2 x_2 + \cdots$$

Then there exists a sequence of elements $y_1, y_2, \ldots \in T_{[x_0]}X$ such that

i. $x_1 = y_1$.
ii. $x_2 = F_2(y_1, y_1) + y_2$.
iii. $x_3 = F_3(y_1, y_1, y_1) + F_2(y_1, y_2) + y_3$.
iv. In general,

$$x_k = \sum_{j=2}^{k} \sum_{\substack{l_1 + 2l_2 + \cdots + f l_f = k \\ l_1 + \cdots + l_f = j}} F_j((y_1)^{l_1}, (y_2)^{l_2}, ..., (y_f)^{l_f}) + y_k.$$

Proof. First note that despite the choices of splittings, each term is well defined because of the lower order terms that appear with it. Also note that ii. is well known in the classical geometry of surfaces, where $F_2 = II$ is the projective *second fundamental form* of X at $[x_0]$. Take adapted coordinates (w^α, z^μ) such that $[x_0] = (0, 0)$ and $T_{[x_0]}X$ is spanned by the first n coordinates $(1 \leq \alpha \leq n)$. Then locally X is given by equations

$$z^\mu = f^\mu(w^\alpha) \tag{1}$$

and

$$F_k(\frac{\partial}{\partial w^{i_1}}, ..., \frac{\partial}{\partial w^{i_k}}) = \sum_\mu \frac{\partial^k f^\mu}{\partial w^{i_1}, ..., \partial w^{i_k}} \frac{\partial}{\partial z^\mu}$$

Now write out the vectors x_j in components, substitute into (1) and compare powers of t. The result follows. □

Let $\tau_{k+1}(X)$ denote the set of all points of the form $[x_0 + x_1 + \cdots + x_k]$, where the x_j are as above, so $\tau_1(X) = X$ and $\tau_2(X)$ is the tangential variety of X, the union of all points on all tangent lines to X. Note that $\tau_k(X)$ is not the k-th osculating variety for $k > 2$, and that the expected dimension of $\tau_k(X)$ is kn. Let $T_{k+1,[x_0]}X$ denote the set of all points of the form $[x_0 + x_1 + \cdots + x_k]$ so $\tau_{k+1}(X) = \cup_{[x_0] \in X} T_{k+1,[x_0]}X$.

3. Components of $\sigma_k(X)$

Let $X \subset \mathbb{P}V$ be a smooth projective variety. Let $\sigma_k^0(X)$ denote the set of points in $\sigma_k(X)\backslash\sigma_{k-1}(X)$ that may be written as the sum of k points on X. Given $[p] \in \sigma_k(X)$, there exist analytic curves $p_1(t), ..., p_k(t)$ in $\hat{X}$, such that for $t \neq 0$, $[p(t)] := [p_1(t) + \cdots + p_k(t)]$ is in $\sigma_k^0(X)$ and $[p(0)] = [p]$. This is true because in the Zariski topology, a set that is open and dense is also dense in the classical topology. If one considers the set of honest secant $\mathbb{P}^{k-1}$'s as a subset of the Grassmanian of all $\mathbb{P}^{k-1}$'s in $\mathbb{P}V$, this set is Zariski open and dense in its Zariski closure and therefore open and dense as a subset of the Zariski closure in the analytic topology.

The point $p(0)$ is in the limiting k-plane corresponding to the first nonvanishing term in the Taylor series for $[p_1(t) \wedge p_2(t) \wedge \cdots \wedge p_k(t)]$, considered as a a point in the Grassmannian $G(k, V) \subset \mathbb{P}(\Lambda^k V)$. When taking a limit, there will be points $q_1, ..., q_s$ such that each of the p_j's limits to one of the q_α's, e.g., $p_1(t), ..., p_{a_1}(t)$ limits to q_1, $p_{a_1+1}(t), ..., p_{a_2}(t)$ limits to q_2 etc... We consider separately the limits L_1 of $p_1(t) \wedge \cdots \wedge p_{a_1}(t)$, etc...

Now assume that $p_i(0)$ is a general point of X for each i. (In particular, this assumption is automatic if X is homogeneous.) We say the limiting k-plane is *standard* if $L_1 \wedge L_2 \wedge \cdots \wedge L_s \neq 0$, i.e., if we may consider the limiting linear spaces associated to each of the q_j's separately. Otherwise we say the limiting k-plane is *exceptional*. An example of an exceptional limit is when there are two limiting points q_1, q_2, but q_2 is in the tangent space of q_1. Another example of an exceptional limit is if at least three points go towards q_1, $q_2 \in T_{3,q_1}X$ and $T_{3,q_1}X$ is not the entire ambient space.

3.1. **Standard components for any variety.** We first consider standard limits, so we may restrict our study to how a curves $p_1(t), ..., p_a(t)$ can limit to a single point x. Write the Taylor series of $p_j(t)$ as

$$p_j(t) = x + x_1^j t + x_2^j t^2 + \cdots$$

Consider first for simplicity the case $a = 3$. The first possible nonzero term is

$$t^2 x \wedge (x_1^2 \wedge x_1^3 - x_1^1 \wedge x_1^3 + x_1^1 \wedge x_1^2).$$

If this term is nonzero then $p \in \sigma_{k-1}(X)$ because any point on such a plane is also on a line of the form $x \wedge v$ with $v \in T_{[x]}X$ and so our three curves only really contribute as two. An easy exercise shows that if this term is zero then either all three terms must be a multiple of one of them, say x_1^1 or two must be equal, say $x_1^2 = x_1^3$. But the second case also leads to $p \in \sigma_{k-1}(X)$ when we examine the t^3 coefficient, so we must have all a multiple of x_1^1. So the only type of term we can have is the span of

$$x, v, II(v, v) + w$$

where $v, w \in T_{[x]}X$. In other words, under the hypotheses that $p \notin \sigma_{k-1}(X)$, the only possible limit is a point of $\tau_3(X)$. Similarly for $a = 4$, the only possible limit is a point of $\tau_4(X)$.

A new phenomenon occurs when $a = 5$. We may obtain a point of $\tau_5(X)$ as above, but a second possibility occurs. Define $\tau_5(X)'$ to be the union of all points in the span of

$$x, x_1, II(x_1, x_1) + x_2, y_1, II(y_1, y_1) + y_2,$$

where $x_1, x_2, y_1, y_2 \in T_x X$. In the notation of above, this will occur if $x_1^1, x_1^2, x_1^3, x_1^4$ span a plane in $T_{[x]}X$.

For the case where all six points limit to the same point there are three possibilities. The first case yields $\tau_6(X)$. For the second, define $\tau_6(X)'$ to be the union of all points in the span of $\langle x, x_1, II(x_1, x_1) + x_2, F_3(x_1, x_1, x_1) + II(x_1, x_2) + x_3, y_1, II(y_1, y_1) + y_2\rangle$ where $x_1, x_2, x_3, y_1, y_2 \in T_x X$. For the third, define $\tau_6(X)''$ to be the union of all points in the span of $\langle x, x_1, II(x_1, x_1) +$

$x_2, F_3(x_1,x_1,x_1)+II(x_1,x_2)+x_3, II(x_1,x_1)+y_2, F_3(x_1,x_1,x_1)+II(x_1,y_2)+y_3\rangle$, where again, the x_j and y_j are points of T_xX.

We now must take the span of s such points. In general, given algebraic varieties $Y_1,...,Y_s \subset \mathbb{P}V$, define their *join* $J(Y_1,...,Y_s) \subset \mathbb{P}V$ to be the Zariski closure of the union of all $\mathbb{P}^{s-1}$'s spanned by points $y_1,...,y_s$ with $y_j \in Y_j$.

From the above discussion, we obtain all standard components of $\sigma_6(X)\backslash\sigma_5(X)$ where $X \subset \mathbb{P}V$ is any variety as follows. (Note that these components will not in general be disjoint.)

(1) $\sigma_6^0(X)$,
(2) $J(\sigma_4(x),\tau_2(X))$,
(3) $J(\sigma_3(X),\tau_3(X))$,
(4) $J(\tau_3(X),\tau_3(X))$,
(5) $J(\sigma_2(X),\tau_2(X),\tau_2(X))$,
(6) $J(\tau_2(X),\tau_2(X),\tau_2(X))$,
(7) $J(\sigma_2(X),\tau_4(X))$,
(8) $J(\tau_2(X),\tau_4(X))$,
(9) $J(X,\tau_2(X),\tau_3(X))$
(10) $J(X,\tau_5(X))$
(11) $J(X,\tau_5(X)')$
(12) $\tau_6(X)$,
(13) $\tau_6(X)'$
(14) $\tau_6(X)''$.

3.2. **Standard components for** $Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$**.** In our case of $X = Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$, X is not only homogenous, but a compact Hermitian symmetric space of rank three. Its only nonzero differential invariants are the second fundamental form $II = F_2$ and the third fundamental form III (see [14], theorem 4.1). The third fundamental form is the component of F_3 taking image in $N_xX/II(S^2T_{[x]}X)$ (see [13], p. 96). Unlike the full F_3, it is a well defined tensor $III \in S^3T_x^*X\otimes(N_xX/II(S^2T_{[x]}X))$. Similar to the situation in lemma 2.1, we choose a splitting of N_xX to make III take values in a subspace of N_xX instead of a quotient space.

Having such simple differential invariants makes it possible to have normal forms for elements of each standard component. We may write any element of X as $p = [a_1\otimes b_1\otimes c_1] = [a_1b_1c_1]$, where all vectors are nonzero. Here and in what follows, the a_j's are elements of A, b_j's of B and c_j's of C, and we omit $\otimes$ in the notation for brevity. Any element of T_pX may be written as $(a_1b_1c_2+a_1b_2c_1+a_2b_1c_1)$, where we allow the possibility of some (but not all) of a_2,b_2,c_2 to be zero.

If $p = [a_1b_1c_1]$ and $v = a_1b_1c_2+a_1b_2c_1+a_2b_1c_1, w = a_1b_1c_3+a_1b_3c_1+a_3b_1c_1, u = a_1b_1c_4+a_1b_4c_1+a_4b_1c_1 \in T_pX$ then, with the obvious choice of splitting;

$$II(v,w) = a_1b_2c_3+a_1b_3c_2+a_2b_1c_3+a_3b_1c_2+a_2b_3c_1+a_3b_2c_1$$

and

$$III(u,v,w) = a_4b_2c_3+a_4b_3c_2+a_2b_4c_3+a_3b_4c_2+a_2b_3c_4+a_3b_2c_4.$$

Here are explicit normal forms for elements of each standard component when $X = Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$. Repetitions and zeros among elements of the sets $\{a_j\}$, $\{b_j\}$ and $\{c_j\}$ are allowed as long as they do not force p into $\sigma_5(X)$.

(1) $p = a_1b_1c_1+\cdots+a_6b_6c_6 \in \sigma_6^0(X)$
(2) $p = a_1b_1c_1+\cdots+a_4b_4c_4+a_5b_5c_5+(a_5b_5c_6+a_5b_6c_5+a_6b_5c_5) \in J(\sigma_4(x),\tau_2(X))$
(3) $p = a_1b_1c_1+a_2b_2c_2+a_3b_3c_3+a_4b_4c_4+(a_4b_4c_5+a_4b_5c_4+a_5b_4c_4)+[(a_4b_4c_6+a_4b_6c_4+a_6b_4c_4)+2(a_4b_5c_5+a_5b_4c_5+a_5b_5c_4)] \in J(\sigma_3(X),\tau_3(X))$

(4) $p = a_1b_1c_1 + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 2(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)] + a_4b_4c_4 + (a_4b_4c_5 + a_4b_5c_4 + a_5b_4c_4) + [(a_4b_4c_6 + a_4b_6c_4 + a_6b_4c_4) + 2(a_4b_5c_5 + a_5b_4c_5 + a_5b_5c_4)] \in J(\tau_3(X), \tau_3(X))$

(5) $p = a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 + (a_3b_3c_4 + a_3b_4c_3 + a_4b_3c_3) + a_5b_5c_5 + (a_5b_5c_6 + a_5b_6c_5 + a_6b_5c_5) \in J(\sigma_2(X), \tau_2(X), \tau_2(X))$

(6) $p = a_1b_1c_1 + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1) + a_3b_3c_3 + (a_3b_3c_4 + a_3b_4c_3 + a_4b_3c_3) + a_5b_5c_5 + (a_5b_5c_6 + a_5b_6c_5 + a_6b_5c_5) \in J(\tau_2(X), \tau_2(X), \tau_2(X))$

(7) $p = a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 + (a_3b_3c_4 + a_3b_4c_3 + a_4b_3c_3) + [(a_3b_3c_5 + a_3b_5c_3 + a_5b_3c_3) + 2(a_3b_4c_4 + a_4b_3c_4 + a_4b_4c_3)] + [(a_3b_3c_6 + a_3b_6c_3 + a_6b_3c_3) + 6a_4b_4c_4 + (a_3b_4c_5 + a_3b_5c_4 + a_4b_3c_5 + a_5b_3c_4 + a_4b_5c_3 + a_5b_4c_3)] \in J(\sigma_2(X), \tau_4(X))$

(8) $p = a_1b_1c_1 + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1) + a_3b_3c_3 + (a_3b_3c_4 + a_3b_4c_3 + a_4b_3c_3) + [(a_3b_3c_5 + a_3b_5c_3 + a_5b_3c_3) + 2(a_3b_4c_4 + a_4b_3c_4 + a_4b_4c_3)] + [(a_3b_3c_6 + a_3b_6c_3 + a_6b_3c_3) + 6a_4b_4c_4 + (a_3b_4c_5 + a_3b_5c_4 + a_4b_3c_5 + a_5b_3c_4 + a_4b_5c_3 + a_5b_4c_3)] \in J(\tau_2(X), \tau_4(X))$

(9) $p = a_1b_1c_1 + a_2b_2c_2 + (a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2) + a_4b_4c_4 + (a_4b_4c_5 + a_4b_5c_4 + a_5b_4c_4) + [(a_4b_4c_6 + a_4b_6c_4 + a_6b_4c_4) + 2(a_4b_5c_5 + a_5b_4c_5 + a_5b_5c_4)] \in J(X, \tau_2(X), \tau_3(X))$

(10) $p = a_1b_1c_1 + a_2b_2c_2 + (a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2) + [(a_2b_2c_4 + a_2b_4c_2 + a_4b_2c_2) + 2(a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2)] + [(a_2b_2c_5 + a_2b_5c_2 + a_5b_2c_2) + 6a_3b_3c_3 + (a_2b_3c_4 + a_2b_4c_3 + a_3b_2c_4 + a_4b_2c_3 + a_3b_4c_2 + a_4b_3c_2)] + [(a_2b_2c_6 + a_2b_6c_2 + a_6b_2c_2) + 2(a_4b_3c_3 + a_3b_4c_3 + a_3b_3c_4) + (a_2b_3c_5 + a_2b_5c_3 + a_3b_2c_5 + a_5b_2c_3 + a_3b_5c_2 + a_5b_3c_2) + 2(a_2b_4c_4 + a_4b_2c_4 + a_4b_4c_2)] \in J(X, \tau_5(X))$

(11) $p = a_1b_1c_1 + a_2b_2c_2 + (a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2) + [(a_2b_2c_4 + a_2b_4c_2 + a_4b_2c_2) + 2(a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2)] + (a_2b_2c_5 + a_2b_5c_2 + a_5b_2c_2) + [(a_2b_2c_6 + a_2b_6c_2 + a_6b_2c_2) + 2(a_2b_5c_5 + a_5b_2c_5 + a_5b_5c_2)] \in J(X, \tau_5(X)')$

(12) $p = a_1b_1c_1 + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 2(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)] + [(a_1b_1c_4 + a_1b_4c_1 + a_4b_1c_1) + 6a_2b_2c_2 + (a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 + a_3b_2c_1)] + [(a_1b_1c_5 + a_1b_5c_1 + a_5b_1c_1) + 2(a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2) + (a_1b_2c_4 + a_1b_4c_2 + a_2b_1c_4 + a_4b_1c_2 + a_2b_4c_1 + a_4b_2c_1) + 2(a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1)] + [(a_1b_1c_6 + a_1b_6c_1 + a_6b_1c_1) + 2(a_2b_2c_4 + a_2b_4c_2 + a_4b_2c_2) + 2(a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2) + (a_1b_2c_5 + a_1b_5c_2 + a_2b_1c_5 + a_5b_1c_2 + a_2b_5c_1 + a_5b_2c_1) + (a_1b_3c_4 + a_1b_4c_3 + a_3b_1c_4 + a_4b_1c_3 + a_3b_4c_1 + a_4b_3c_1)] \in \tau_6(X)$

(13) $p = a_1b_1c_1 + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 2(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)] + [(a_1b_1c_4 + a_1b_4c_1 + a_4b_1c_1) + 6a_2b_2c_2 + (a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 + a_3b_2c_1)] + (a_1b_1c_5 + a_1b_5c_1 + a_5b_1c_1) + [(a_1b_1c_6 + a_1b_6c_1 + a_6b_1c_1) + 2(a_1b_5c_5 + a_5b_1c_5 + a_5b_5c_1)] \in \tau_6(X)'$

(14) $p = a_1b_1c_1 + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 2(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)] + [(a_1b_1c_4 + a_1b_4c_1 + a_4b_1c_1) + 6a_2b_2c_2 + (a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 + a_3b_2c_1)] + [(a_1b_1c_5 + a_1b_5c_1 + a_5b_1c_1) + 2(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)] + [(a_1b_1c_6 + a_1b_6c_1 + a_6b_1c_1) + 6a_2b_2c_2 + (a_1b_2c_5 + a_1b_5c_2 + a_2b_1c_5 + a_5b_1c_2 + a_2b_5c_1 + a_5b_2c_1)] \in \tau_6(X)''$

3.3. **Exceptional components of** $\sigma_6(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$**.** For an arbitrary variety, writing down all possible exceptional components is an intractible problem, but, again by the simplicity of the Segre, there are only two exceptional components of $\sigma_6(X) \backslash \sigma_5(X)$ and both occur in the case of two limit points.

To see this, say there are two exceptional limit points p, q and all six curves limit to these. *A priori* the possible exceptional positions of p and q are that $p \in T_qX$ or that $p \in T_{k,q}X$, for some $k > 1$. However, it is easy to see that an element of $T_{k,q}X$ cannot be decomposable unless either the "new" vector (denoted y_k in the lemma) is zero, in which case we are reduced to a point of $\sigma_5(X)$, or all the Fubini forms occuring in the k-th term are zero and y_k is decomposable - but in this case the k-th term is just a point of T_qX and no new phenomenon occurs.

So say $p \in T_qX$. *A priori* there could be five different types of limits, depending on the number of curves that limit to p and the number that limit to q. But $p \in T_qX$ implies $q \in T_pX$, so by symmetry we are reduced to three cases, five points limiting to p and one to q, four to p and two to q, and three to each p and q. Without loss of generality take $p = a_1b_1c_1$ and $q = a_1b_1c_2 \in T_pX$. If the x_1 term in the expansion for p is not equal to q, then nothing new can occur as the expansions won't interfere with each other (in fact one ends up with a point of $\sigma_5(X)$).

In all cases, there is no ambiguity as to which terms in the Taylor expansions for p and q must contribute, as there is a unique choice of terms to wedge together that yield a term of lowest order.

Consider the case of 5 points limiting to p and one to q. In order to get something new we must have the first tangent vector to p be $a_1b_1c_2$, as then we can use the first order term in the expansion of q. In this case we get a point of the following form, where 0's have been included where terms that ordinarily would not be zero are, e.g., the first zero represents $0 = II(a_1b_1c_2, a_1b_1c_2)$. Also for clarity a redundant $a_1b_1c_2$ is included in double parentheses.

$$\begin{aligned} x =& a_1b_1c_1 + (a_1b_1c_2) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 0] \\ &+ [(a_1b_1c_4 + a_1b_4c_1 + a_4b_1c_1) + 0 + (a_1b_3c_2 + a_3b_1c_2)] \\ &+ [(a_1b_1c_5 + a_1b_5c_1 + a_5b_1c_1) + 0 + (a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1) + (a_1b_4c_2 + a_4b_1c_2)] \\ &+ ((a_1b_1c_2)) + (a_1b_1c_6 + a_1b_6c_2 + a_6b_1c_2). \end{aligned}$$

Here $x \in \tau_5(X) \subset \sigma_5(X)$, which can be seen by making the following substitutions: $\tilde{c}_5 = c_5 + c_6$, $\tilde{b}_4 = b_4 + b_6$, $\tilde{b}_5 = b_5 - b_6$, $\tilde{a}_4 = a_4 + a_6$, $\tilde{a}_5 = a_5 - a_6$.

If the limit to p is in $\tau_5(X)'$ we obtain

$$\begin{aligned} x =& a_1b_1c_1 + (a_1b_1c_2) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 0] \\ &+ (a_1b_1c_4 + a_1b_4c_1 + a_4b_1c_1) \\ &+ [(a_1b_1c_5 + a_1b_5c_1 + a_5b_1c_1) + (a_1b_4c_4 + a_4b_1c_4 + a_4b_4c_1)] \\ &+ ((a_1b_1c_2)) + (a_1b_1c_6 + a_1b_6c_2 + a_6b_1c_2). \end{aligned}$$

Here we may set $\tilde{c}_5 = c_5 + c_3$, $\tilde{b}_5 = b_5 + b_3$, $\tilde{a}_5 = a_5 + a_3$ to see that $x \in \sigma_5(X)$.

Still another possibility exists if in the cases above, the tangent vector to q, namely $(a_1b_1c_6 + a_1b_6c_2 + a_6b_1c_2)$ already appears as one of the terms in the expansion for p, in which case we get to examine another term in the Taylor series for q. This can happen when the limit to p is a point of $\tau_5(X)$ and we have the coincidences $c_6 = c_4$, $b_6 = b_3$, $a_6 = a_3$, $a_4, b_4 = 0$. Under these circumstances we get

$$\begin{aligned} x =& a_1b_1c_1 + (a_1b_1c_2) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 0] \\ &+ [(a_1b_1c_4) + 0 + (a_1b_3c_2 + a_3b_1c_2)] \\ &+ [(a_1b_1c_5 + a_1b_5c_1 + a_5b_1c_1) + 0 + (a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1) + 0] \\ &+ ((a_1b_1c_2)) + ((a_1b_1c_4 + a_1b_3c_2 + a_3b_1c_2)) \\ &+ [(a_7b_1c_2 + a_1b_7c_2 + a_1b_1c_7) + (a_1b_3c_4 + a_3b_1c_4 + a_3b_3c_2)]. \end{aligned}$$

Call this case EX_1. If the limit to p is a point of $\tau_5(X)'$ there is no comparable term to cancel the term in the expansion for q.

One could try to make the $[(a_7b_1c_2 + a_1b_7c_2 + a_1b_1c_7) + (a_1b_3c_4 + a_3b_1c_4 + a_3b_3c_2)]$ term also appear in the limit for p, but this again forces too much degeneracy.

We have now examined all possibilities for 5 curves coming together to one point.

Now say the split between p and q is four/two. Again, if q is not the initial tangent vector to p we can get nothing new, and if it is, we obtain:

$$\begin{aligned} x =& a_1b_1c_1 + (a_1b_1c_2) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 0] \\ &+ [(a_1b_1c_4 + a_1b_4c_1 + a_4b_1c_1) + 0 + (a_1b_3c_2 + a_3b_1c_2)] \\ &+ ((a_1b_1c_2)) + (a_1b_1c_5 + a_1b_5c_2 + a_5b_1c_2) \\ &+ [(a_1b_1c_6 + a_1b_6c_2 + a_6b_1c_2) + 2(a_1b_5c_5 + a_5b_1c_5 + a_5b_5c_2)]. \end{aligned}$$

Call this case EX_2.

Now consider the case where the tangent vector to q occurs in the expansion for p:

$$\begin{aligned} x =& a_1b_1c_1 + (a_1b_1c_2) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 0] \\ &+ [(a_1b_1c_4) + 0 + (a_1b_3c_2 + a_3b_1c_2)] \\ &+ ((a_1b_1c_2)) + ((a_1b_1c_3 + a_1b_3c_2 + a_3b_1c_2)) \\ &+ [(a_1b_1c_5 + a_1b_5c_2 + a_5b_1c_2) + 2(a_1b_3c_2 + a_3b_1c_4 + a_3b_3c_2)] \\ &+ [a_1b_1c_6 + a_1b_6 + a_1b_6c_2 + a_6b_1c_2 + a_4b_3c_4 + a_1b_5c_4 + a_5b_1c_4 + a_1b_3c_5 + a_3b_5c_2 + a_5b_3c_2]. \end{aligned}$$

Set $\tilde{c}_6 = c_6 + c_4$, $\tilde{c}_5 = c_5 + c_2$, and we see that $x \in \sigma_5(X)$. Higher order dengerations are similarly eliminated.

Finally consider a 3-3 split. Assuming q is the first tangent vector to p in the expansion, we get

$$\begin{aligned} x =& a_1b_1c_1 + ((a_1b_1c_2)) + [(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) + 0] \\ &+ [(a_1b_1c_4 + a_1b_4c_1 + a_4b_1c_1) + (a_1b_3c_2 + a_3b_1c_2)] \\ &+ (a_1b_1c_2) + (a_1b_1c_5 + a_1b_5c_2 + a_5b_1c_2) \\ &+ [(a_1b_1c_6 + a_1b_6c_2 + a_6b_1c_2) + 2(a_1b_5c_5 + a_5b_1c_5 + a_5b_5c_2)] \end{aligned}$$

Let $\tilde{b}_3 = b_3 + b_4$, $\tilde{b}_6 = b_6 + \tilde{b}_3$, $\tilde{a}_3 = a_3 + a_4$, $\tilde{a}_6 = a_6 + \tilde{a}_3$, and $\tilde{c}_6 = c_6 + c_4$ and we see $x \in \sigma_5(X)$.

Assuming further coincidences similarly yields nothing new.

4. Case by case arguments

We will use variants of the proof of Proposition (17.9) in [7], which is due to Baur. Since there is a misprint in the proof in [7] (a prohibited re-ordering of indices), we reproduce a proof here.

Theorem 4.1. *If A is a simple k-algebra then the rank of the multiplication operator M_A is $\geq 2 dim\, A - 1$.*

Proof. Let $n := \dim A$ and express M_A optimally as $M_A = \alpha^1 \otimes \beta^1 \otimes c_1 + \cdots + \alpha^r \otimes \beta^r \otimes c_r$ and assume, to obtain a contradiction, that $r < 2\dim A - 1$. (We switch notation, working in $A^* \otimes B^* \otimes C$, using α^i to denote elements of A^*, and β^i to denote elements of $B^* = A^*$.)

By reordering if necessary, we may assume $\alpha^1, ..., \alpha^n$ is a basis of A^*. Let $b \in \langle \beta^n, \beta^{n+1}, ..., \beta^r \rangle^\perp$ be a nonzero element and consider the left ideal Ab. We have $Ab \subseteq \langle c_1, ..., c_{n-1} \rangle$. Let $L \supseteq Ab$ be a maximal left ideal containing Ab and let $n - m = \dim L$. Since $\dim Ab \geq m$ (the minimal dimension of an ideal), at least m of the $\beta^1(b), ..., \beta^{n-1}(b)$ are nonzero (using again the linear independence of $\alpha^1, ..., \alpha^n$). Reorder among $1, ..., n-1$ such that the first m are nonzero, we have $\langle c_1, ..., c_m \rangle \subseteq Ab$ (by the linear independence of $\alpha^1, ..., \alpha^m$).

Note that $\beta^m, ..., \beta^r$ span A^* (otherwise let $z \in \langle \beta^m, ..., \beta^r \rangle^\perp$ and consider the left ideal Az which is too small). In particular, restricted to L, a subset of them spans L^*. We already know $\beta^m\,|_L \neq 0$ so we use that as a first basis vector. Let $\beta^{i_1}, ..., \beta^{i_{n-m-1}}$ be a subset of $\beta^{m+1}, ..., \beta^r$ such that together with β^m, when restricted to L they form a basis of L^*. Now

let $j_1,...,j_{r-n}$ be a complementary set of indices such that $\{i_1,...,i_{n-m-1}\} \cup \{j_1,...,j_{r-n}\} = \{m+1,...,r\}$ and take $a \in \langle \alpha^{j_1},...,\alpha^{j_{r-n}}\rangle^\perp$ a nonzero element. Consider the right ideal $aA \subset \langle c_1,...,c_{m-1},c_m,c_{i_1},...,c_{i_{n-m-1}}\rangle$. For any $y \in A$, there exists $w \in L$ such that $\beta^m(y) = \beta^m(w)$ and $\beta^{i_f}(y) = \beta^{i_f}(w)$ for $1 \le f \le n-m-1$, so $ay - aw \in \langle c_1,...,c_{m-1}\rangle$. But since aw and the right hand side are both included in L, we conclude $aA \subseteq L$ which is a contradiction as a left ideal cannot contain a nontrivial right ideal. $\square$

We now specialize to matrix multiplication of 2×2 matrices which we denote by $MMult$. Let $X = Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$. The theorem above implies $MMult \notin \sigma_6^0(X)$. We now show it is not in any of the other possiblilities. Let A denote the algebra of two by two matrices. Note that all ideals of A must be of dimension two.

Proposition 4.2. *$Mmult \notin J(\sigma_4(X), \tau_2(X))$, $J(\sigma_2(X), \tau_2(X), \tau_2(X))$ or $J(\tau_2(X), \tau_2(X), \tau_2(X))$.*

Proof. Say otherwise, that we had an expression respectively

$$\begin{aligned}
MMult =& \alpha^1\beta^1c_1 + \cdots + \alpha^5\beta^5c_5 + (\alpha^5\beta^5c_6 + \alpha^5\beta^6c_5 + \alpha^6\beta^5c_5)\\
MMult =& \alpha^1\beta^1c_1 + \alpha^2\beta^2c_2 + \alpha^3\beta^3c_3 + (\alpha^3\beta^3c_4 + \alpha^3\beta^4c_3 + \alpha^4\beta^3c_3) + \alpha^5\beta^5c_5\\
&+ (\alpha^5\beta^5c_6 + \alpha^5\beta^6c_5 + \alpha^6\beta^5c_5)\\
MMult =& \alpha^1\beta^1c_1 + (\alpha^1\beta^1c_2 + \alpha^1\beta^2c_1 + \alpha^2\beta^1c_1) + \alpha^3\beta^3c_3 + (\alpha^3\beta^3c_4 + \alpha^3\beta^3c_4 + \alpha^4\beta^3c_3)\\
&+ \alpha^5\beta^5c_5 + (\alpha^5\beta^5c_6 + \alpha^5\beta^6c_5 + \alpha^6\beta^5c_5)
\end{aligned}$$

which we refer to as the first, second and third cases.

We first claim that $\alpha^1,\alpha^2,\alpha^3,\alpha^4$ or $\beta^1,\beta^2,\beta^3,\beta^4$ must be linearly independent. Otherwise, say both sets were dependent. Let $a' \in \langle\alpha^1,\alpha^2,\alpha^3,\alpha^4\rangle^\perp$, $b' \in \langle\beta^1,\beta^2,\beta^3,\beta^4\rangle^\perp$, we have $a'A = Ab' = \langle c_5, c_6\rangle$, a contradiction. The same conclusion holds for the 4-ples of vectors with indices $1,2,5,6$, and those with indices $3,4,5,6$. (Here and in all arguments that follow, when we talk about finding vectors like a, a', b, b' etc..., we mean nonzero vectors.)

We also note various independences among the c_j's. In case 3, c_1, c_3, c_5 must be independent as otherwise consider $a \in \langle\alpha^1,\alpha^3,\alpha^5\rangle^\perp$, $b \in \langle\beta^1,\beta^3,\beta^5\rangle^\perp$. We have $aA \subseteq \langle c_1,c_3,c_5\rangle$ and in fact equality by the linear dependence, but similarly for Ab, thus $aA = Ab$, a contradiction. In cases 1 and 2, c_3, c_5, c_6 must be independent as otherwise in case 1 we could consider $a \in \langle\alpha^1,\alpha^2,\alpha^4\rangle^\perp$ and in case 2, $a \in \langle\alpha^1,\alpha^2,\alpha^3\rangle^\perp$. In both cases we get $aA = \langle c_3,c_5,c_6\rangle$ but taking a corresponding b yields a contradiction as above.

Cases 1 and 2: without loss of generality we assume $\alpha^3,\alpha^4,\alpha^5,\alpha^6$ are independent and consider $b \in \langle\beta^1,\beta^2,\beta^4\rangle^\perp$ in case 1, and $b \in \langle\beta^1,\beta^2,\beta^3\rangle^\perp$ in case 2. In both cases $Ab \subset \langle c_3,c_5,c_6\rangle$. In case 2 we have the following matrix mapping the coefficients of $\alpha^3,\alpha^5,\alpha^6$ to the coefficients of c_3, c_5, c_6:

$$Mult(\cdot, b) = \begin{pmatrix} \beta^4(b) & 0 & 0\\ 0 & \beta^5(b)+\beta^6(b) & \beta^5(b)\\ 0 & \beta^5(b) & 0\end{pmatrix}$$

and in case 1 the matrix is the same except $\beta^3(b)$ replaces $\beta^4(b)$. By the linear indepence of c_3, c_5, c_6 and $\alpha^3,\alpha^5,\alpha^6$, the matrix must have rank two.

There are two subcases to consider depending on whether or not $\beta^3(b) = 0$ in case 1 (resp. $\beta^4(b) = 0$ in case 2).

Subcase 1: If $\beta^3(b) = 0$ (resp. $\beta^4(b) = 0$) then $\beta^5(b) \neq 0$ and $Ab = \langle c_5, c_6\rangle$. We claim that $\beta^1,\beta^2,\beta^4,\beta^5$ (resp. $\beta^1,\beta^2,\beta^3,\beta^5$) is a basis of A^* as otherwise let $b' \in \langle\beta^1,\beta^2,\beta^4,\beta^5\rangle^\perp$ (resp. $b' \in \langle\beta^1,\beta^2,\beta^3,\beta^5\rangle^\perp$). We would have $Ab' = \langle c_3, c_5\rangle$ which has a one-dimensional intersection

with Ab, thus a contradiction. Thus at least one of β^1,β^2,β^4 (resp. β^1,β^2,β^3) together with β^5 restricted to $(Ab)^*$ forms a basis of $(Ab)^*$.

Say β^1,β^5 gives the basis. Then take $a\in\langle\alpha^2,\alpha^3,\alpha^4\rangle^\perp$, we have $aA\subset\langle c_1,c_5,c_6\rangle$. But now for any $y\in A$ there exists $w\in Ab$ such that $\beta^1(y)=\beta^1(w)$. Consider $ay-aw\subset\langle c_5,c_6\rangle=Ab$. Since $aw\in Ab$ we conclude $ay\in Ab$ and thus $aA\subseteq Ab$, a contradiction. The argument is the same if β^2,β^5 gives the basis, just change indices. In case 1, the argument is still the same if β^4,β^5 gives the basis, again just change indices.

Say β^3,β^5 gives the basis in case 2. Then take $a\in\langle\alpha^1,\alpha^2,\alpha^3\rangle^\perp$ so $aA\subseteq\langle c_3,c_5,c_6\rangle$. Now use β^3 to show $aA\subseteq\langle c_5,c_6\rangle=Ab$ to again obtain a contradiction.

Subcase 2: If $\beta^3(b)\neq 0$ (resp. $\beta^4(b)\neq 0$) then $\beta^5(b)=0$ and $\beta^6(b)\neq 0$ and $Ab=\langle c_3,c_5\rangle$. We claim that $\beta^1,\beta^2,\beta^3,\beta^4$ is a basis of A^* as otherwise let $b'\in\langle\beta^1,\beta^2,\beta^3,\beta^4\rangle^\perp$ to get a notrivial intersection $Ab\cap Ab'$ and a contradiction. Thus at least one of β^1,β^2,β^4 (resp. β^1,β^2,β^3) together with β^3 (resp. β^4) restricted to $(Ab)^*$ forms a basis of $(Ab)^*$. Each case leads to a contradiction as in the first subcase, finishing the proof for cases 1 and 2.

Case 3: We first claim that $\alpha^1,\alpha^3,\alpha^5$ are linearly independent. Otherwise consider $a\in\langle\alpha^1,\alpha^3,\alpha^5,\alpha^6\rangle^\perp$ and $a'\in\langle\alpha^1,\alpha^3,\alpha^4,\alpha^5\rangle^\perp$. We would have $aA=\langle c_1,c_3\rangle$, $a'A=\langle c_1,c_5\rangle$ with a nontrivial intersection and thus a contradiction.

Now consider $b\in\langle\beta^1,\beta^3,\beta^5\rangle^\perp$, so $Ab\subset\langle c_1,c_3,c_5\rangle$. We have the matrix

$$Mult(\cdot,b)=\begin{pmatrix}\beta^2(b)&0&0\\0&\beta^4(b)&0\\0&0&\beta^6(b)\end{pmatrix}$$

which must have rank two by the linear independence of c_1,c_3,c_5 and $\alpha^1,\alpha^3,\alpha^5$. By symmetry we may assume $\beta^6(b)=0$ so $Ab=\langle c_1,c_3\rangle$ and $\beta^1,\beta^3,\beta^5,\beta^6$ are linearly dependent. We claim that $\beta^1,\beta^2,\beta^3,\beta^5$ gives a basis of A^* as otherwise we could find a $b'\in\langle\beta^1,\beta^2,\beta^3,\beta^5,\beta^6\rangle^\perp$ with $Ab'=\langle c_3\rangle$, a contradiction. Thus at least one of β^1,β^3,β^5 restricted to $(Ab)^*$ together with β^2 gives a basis of $(Ab)^*$.

Say β^1,β^2 gives a basis. Then take $a\in\langle\alpha^3,\alpha^5,\alpha^6\rangle^\perp$ so $aA\subset\langle c_1,c_2,c_3\rangle$. But c_2 appears in $MMult$ with coefficient $\alpha^1\beta^1$ so we may argue as above to see $aA=\langle c_1,c_3\rangle=Ab$ to obtain a contradiction. Similarly for β^3,β^2, using $a\in\langle\alpha^1,\alpha^5,\alpha^6\rangle^\perp$ and the case of β^5,β^2 using $a\in\langle\alpha^1,\alpha^3,\alpha^5\rangle^\perp$. This concludes the proof in case 3.

□

Proposition 4.3. *$MMult\notin J(\sigma_3(X),\tau_3(X)),J(\tau_3(X),\tau_3(X)),J(X,\tau_2(X),\tau_3(X))$.*

Proof. Assume otherwise that

$$\begin{aligned}MMult=&\alpha^1\beta^1c_1+\alpha^2\beta^2c_2+\alpha^3\beta^3c_3+\alpha^4\beta^4c_4+(\alpha^4\beta^4c_5+\alpha^4\beta^5c_4+\alpha^5\beta^4c_4)\\&+(\alpha^4\beta^4c_6+\alpha^4\beta^6c_4+\alpha^6\beta^4c_4)+2(\alpha^4\beta^5c_5+\alpha^5\beta^4c_5+\alpha^5\beta^5c_4)\end{aligned}$$

for the first case and similarly for the other two cases. Let $b\in\langle\beta^1,\beta^2,\beta^3\rangle^\perp$ so $Ab\subseteq\langle c_4,c_5,c_6\rangle$. Note that c_4,c_5,c_6 must be linearly independent, as otherwise take $a\in\langle\alpha^1,\alpha^2,\alpha^3\rangle^\perp$ and $aA=Ab$.

Now consider the linear map $MMult(\cdot,b)$. Assume for the moment that $\alpha^4,\alpha^5,\alpha^6$ are linearly independent. With respect to bases c_4,c_5,c_6 and $\alpha^4,\alpha^5,\alpha^6$, the map $MMult(\cdot,b)$ has matrix

$$\begin{pmatrix}\beta^4(b)+\beta^5(b)+\beta^6(b)&\beta^4(b)+\beta^5(b)&\beta^4(b)\\\beta^4(b)+\beta^5(b)&\beta^4(b)&0\\\beta^4(b)&0&0\end{pmatrix}$$

It must have two-dimensional image, but this can occur only if $\beta^4(b) = 0$, so we conclude $Ab = \langle c_4, c_5\rangle$. But now take $a \in \langle \alpha^1, \alpha^2, \alpha^3\rangle^\perp$ and the same argument, assuming $\beta^4, \beta^5, \beta^6$ are linearly independent, gives $aA = \langle c_4, c_5\rangle$, a contradiction.

Now say $\beta^4, \beta^5, \beta^6$ fail to be linearly independent and consider the family of ideals Ab' one obtains as b' ranges over $\langle \beta^4, \beta^5, \beta^6\rangle^\perp$. These ideals must all be contained in $\langle c_1, c_2, c_3\rangle$ hence they must be constant, otherwise we would have two left ideals with a nontrivial intersection. But that means either c_1, c_2, c_3 fail to be linearly independent, which gives a contradiction as usual, or $\alpha^1, \alpha^2, \alpha^3$ fail to be linearly independent. But now if $\alpha^1, \alpha^2, \alpha^3$ fail to be linearly independent, consider $a' \in \langle \alpha^1, \alpha^2, \alpha^3, \alpha^4\rangle^\perp$, we have $a'A = \langle c_4, c_5\rangle$ equaling Ab above and giving a contradiction unless $\alpha^4, \alpha^5, \alpha^6$ also fail to be linearly independent.

Finally, assuming both $\alpha^4, \alpha^5, \alpha^6$ and $\beta^4, \beta^5, \beta^6$ are dependent, consider, in the first two cases $\tilde a \in \langle \alpha^3, \alpha^4, \alpha^5, \alpha^6\rangle^\perp$ and $\tilde b \in \langle \beta^3, \beta^4, \beta^5, \beta^6\rangle^\perp$ and in the third case $\tilde a \in \langle \alpha^2, \alpha^4, \alpha^5, \alpha^6\rangle^\perp$ and $\tilde b \in \langle \beta^2, \beta^4, \beta^5, \beta^6\rangle^\perp$. In all cases we have $\tilde aA = A\tilde b = \langle c_1, c_2\rangle$ and thus a contradiction. □

Proposition 4.4. *$MMult \notin J(\sigma_2(X), \tau_4(X)), J(\tau_2(X), \tau_4(X))$.*

Proof. We use normal forms as in §3. Consider $b \in \langle \beta^1, \beta^2, \beta^3\rangle^\perp$, so $Ab \subset \langle c_3, c_4, c_5\rangle$. Assuming $\alpha^3, \alpha^4, \alpha^5$ are linearly independent, we get the same type of matrix as above and conclude $\beta^4(b) = 0$ and $Ab = \langle c_3, c_4\rangle$. If we also assume $\beta^3, \beta^4, \beta^5$ are linearly independent, taking $a \in \langle \alpha^1, \alpha^2, \alpha^3\rangle^\perp$ we have $aA = Ab$ a contradiction.

If both $\alpha^3, \alpha^4, \alpha^5$ and $\beta^3, \beta^4, \beta^5$ are linearly dependent, we may take $a' \in \langle \alpha^3, \alpha^4, \alpha^5, \alpha^6\rangle^\perp$ and $b' \in \langle \beta^3, \beta^4, \beta^5, \beta^6\rangle^\perp$ to get $a'A = Ab' = \langle c_1, c_2\rangle$.

So assume $\beta^3, \beta^4, \beta^5$ are linearly dependent and $\alpha^3, \alpha^4, \alpha^5$ are independent. Consider the family of elements $b'' \in \langle \beta^3, \beta^4, \beta^5\rangle^\perp$ so $Ab'' \subset \langle c_1, c_2, c_3\rangle$ this gives a family of left ideals in $\langle c_1, c_2, c_3\rangle$ but as above there must be just one ideal, which must be $\langle c_1, c_2\rangle$ as this is what one obtains for the element in $\langle \beta^3, \beta^4, \beta^5, \beta^6\rangle^\perp$. But this implies $\beta^6(b'') = 0$ for all $b'' \in \langle \beta^3, \beta^4, \beta^5\rangle^\perp$, i.e., $\beta^6 \subset \langle \beta^3, \beta^4, \beta^5\rangle$ so now we may take, in the first case of the proposition $b_0 \in \langle \beta^2, \beta^3, \beta^4, \beta^5, \beta^6\rangle^\perp$ and $b_0 \in \langle \beta^1, \beta^3, \beta^4, \beta^5, \beta^6\rangle^\perp$ in the second to obtain a one-dimensional left ideal Ab_0. □

Proposition 4.5. *$MMult \notin J(X, \tau_5(X))$.*

Proof. Let $b \in \langle \beta^1, \beta^2, \beta^3\rangle^\perp$ thus $Ab \subseteq \langle c_2, c_3, c_4\rangle$. As usual we must have c_2, c_3, c_4 linearly independent. Considering the linear map $MMult(\cdot, b)$, the only α^j's that arise are $\alpha^2, \alpha^3, \alpha^4$ and, assuming they are linearly independent, the 3×3 matrix will have zero determinant only if $\beta^4(b) = 0$, but then $Ab = \langle c_1, c_2\rangle$.

Now consider $a \in \langle \alpha^1, \alpha^2, \alpha^3\rangle^\perp$, we see if $\beta^2, \beta^3, \beta^4$ are linearly independent we have $aA = Ab$ a contradiction.

If say $\beta^2, \beta^3, \beta^4$ fail to be linearly independent, consider $b' \in \langle \beta^2, \beta^3, \beta^4, \beta^5\rangle^\perp$ so $Ab' = \langle c_1, c_2\rangle$. Now consider $b'' \in \langle \beta^2, \beta^3, \beta^4, \beta^6\rangle^\perp$. We have $Ab'' \subset \langle c_1, c_2, c_3\rangle$ so it has a nonzero intersection with Ab', hence a contradiction unless it equals Ab'. But examining the matrix, even without assuming independence of $\alpha^1, \alpha^2, \alpha^3$, the only way $Ab'' = Ab'$ is if $\beta^5 \subset \langle \beta^2, \beta^3, \beta^4, \beta^6\rangle$, and if this occurs we simply take $\tilde b \subset \langle \beta^2, \beta^3, \beta^4, \beta^5, \beta^6\rangle^\perp$ to obtain a one dimensional ideal $A\tilde b = \langle c_1\rangle$. □

Proposition 4.6. *$MMult \notin \tau_6(X)$*

Proof. Let $b \in \langle \beta^1, \beta^2, \beta^3\rangle^\perp$ and consider the ideal $Ab \subset \langle c_1, c_2, c_3\rangle$. Again, c_1, c_2, c_3 must be linearly independent. If $\alpha^1, \alpha^2, \alpha^3$ are also linearly independent, then the only way for Ab to be two dimensional is if $\beta^4(b) = 0$. Thus $Ab = \langle c_1, c_2\rangle$. The same reasoning applied to $a \in \langle \alpha^1, \alpha^2, \alpha^3\rangle^\perp$ implies $aA = \langle c_1, c_2\rangle$, a contradiction, so at least one of the sets $\alpha^1, \alpha^2, \alpha^3$, $\beta^1, \beta^2, \beta^3$ must be linearly dependent.

Say just one set, e.g., $\beta^1, \beta^2, \beta^3$ fails to be linearly independent, then by the reasoning above $\beta^4 \subset \langle \beta^1, \beta^2, \beta^3\rangle$ because $\beta^4(b) = 0$ for all $b \in \langle \beta^1, \beta^2, \beta^3\rangle^\perp$. But then there exists $b' \in \langle \beta^1, \beta^2, \beta^3, \beta^4, \beta^5\rangle^\perp$ yielding an ideal $Ab' = \langle c_2\rangle$, a contradiction.

Finally say both sets are dependent. Consider $b'' \in \langle \beta^1,\beta^2,\beta^3,\beta^4\rangle^{\perp}$ and $a'' \in \langle \alpha^1,\alpha^2,\alpha^3,\alpha^4\rangle^{\perp}$. We have $Ab'' = a''A = \langle c_1, c_2\rangle$, a contradiction. □

Proposition 4.7. *$MMult \notin J(X,\tau_5(X)'),\tau_6(X)',\tau_6(X)'', EX_1, EX_2$*

Proof. Consider $b \in \langle \beta^1,\beta^2,\beta^3\rangle^{\perp}$ in the first four cases and $b \in \langle \beta^1,\beta^3,\beta^5\rangle^{\perp}$ for the last. In all cases Ab is a fixed two dimensional ideal (e.g. $\langle c_2, c_5\rangle$ in the first) but taking $a \in \langle \alpha^1,\alpha^2,\alpha^3\rangle^{\perp}$ in the first four cases and $a \in \langle \alpha^1,\alpha^3,\alpha^5\rangle^{\perp}$ for the last yields the same two dimensional ideal, hence a contradiction. □

5. Cases overlooked in the original article

Let $X \subset \mathbb{P}V$ be a projective variety and let $p \in \sigma_r(X)$. Then there exist curves $x_1(t),...,x_r(t) \subset \hat{X}$ with $p \in \lim_{t\to 0}\langle x_1(t),...,x_r(t)\rangle$. We are interested in the case when $\dim\langle x_1(0),...,x_r(0)\rangle < r$. In [1] it was mistakenly asserted that the only way for this to happen was for some of the points to coincide. In what follows I show that the cases I neglected to account for also cannot be matrix multiplication, filling the gap in the proof.

Use the notation $x_j = x_j(0)$. Assume for the moment that $x_1,...,x_{r-1}$ are linearly independent. Then we may write $x_r = c_1x_1 + \cdots + c_{r-1}x_{r-1}$ for some constants $c_1,...,c_{r-1}$. Write each curve $x_j(t) = x_j + tx_j' + t^2x_j'' + \cdots$ where derivatives are taken at $t = 0$.

Consider the Taylor series

$$\begin{aligned}
&x_1(t)\wedge\cdots\wedge x_r(t) = \\
&(x_1 + tx_1' + t^2x_1'' + \cdots)\wedge\cdots\wedge(x_{r-1} + tx_{r-1}' + t^2x_{r-1}'' + \cdots)\wedge(x_r + tx_r' + t^2x_r'' + \cdots) \\
&\quad = t((-1)^r(c_1x_1' + \cdots c_{r-1}x_{r-1}' - x_r')\wedge x_1\wedge\cdots\wedge x_{r-1}) + t^2(...) + \cdots
\end{aligned}$$

If the t coefficient is nonzero, then p lies in the linear span of $x_1,...,x_{r-1},(c_1x_1' + \cdots c_{r-1}x_{r-1}' - x_r')$.

If the t coefficient is zero, we have $c_1x_1' + \cdots + c_{r-1}x_{r-1}' - x_r' = e_1x_1 + \cdots e_{r-1}x_{r-1}$ for some constants $e_1,...,e_{r-1}$. In this case we must examine the t^2 coefficient of the expansion. It is

$$\left(\sum_{k=1}^{r-1} e_kx_k' + \sum_{j=1}^{r-1} c_jx_j'' - x_r''\right)\wedge x_1\wedge\cdots\wedge x_{r-1}$$

One continues to higher order terms if this is zero.

More generally, if only $x_1,...,x_p$ are linearly independent, use index ranges $1 \le j \le p$, $p+1 \le s \le r$ and write $x_s = c_s^jx_j$ (summation convention), then the first possible nonzero term in the Taylor series is

$$x_1\wedge\cdots\wedge x_p\wedge(c_{p+1}^1x_1' + \cdots + c_{p+1}^px_p' - x_{p+1}')\wedge\cdots\wedge(c_r^1x_1' + \cdots + c_r^px_p' - x_r')$$

which (up to a sign) is the coefficient of t^{r-p}.

Examples of such limits in the case of the Segre variety. We look for linear subspaces $L^s \subset A_1\otimes\cdots\otimes A_n$ such that $\#(\mathbb{P}L \cap Seg(\mathbb{P}A_1\times\cdots\times\mathbb{P}A_n)) > s$. Let $v_d(\mathbb{P}V) \subset \mathbb{P}S^dV$ denote the d-th Veronese. Consider

$$\langle v_{d_1}(\mathbb{P}^{i_1-1})\times\cdots\times v_{d_k}(\mathbb{P}^{i_k-1})\rangle \tag{2}$$

where we fix embeddings

$$S^{d_j}\mathbb{C}^{i_j} \subset (\mathbb{C}^{i_j})^{\otimes d_j} \subset A_{j_1}\otimes\cdots\otimes A_{j_{d_j}}$$

with $d_1 + \cdots d_k = n$ and

$$s \le \binom{i_1+d_1-1}{d_1}\cdots\binom{i_k+d_k-1}{d_k}$$

If strict inequality holds, we take a linear section of (2) to get an s-dimensional linear space.

For example, if we take each $d_j = 1$, we are just choosing linear subspaces $A'_j \subset A_j$ of dimensions a'_j such that $a'_1 \cdots a'_n = s$.

There are other examples, e.g., $\mathbb{P}^4 \cap Seg(\mathbb{P}^2 \times \mathbb{P}^2)$ is six ($= deg(Seg(\mathbb{P}^2 \times \mathbb{P}^2))$) points for a generic $\mathbb{P}^4$.

Conditions for the t coefficient to be zero in the 3-factor Segre case. Write $x_r = \xi^j x_j$, and assume $x_1, ..., x_{r-1}$ are linearly independent. Let $x_\alpha = \tilde{a}_\alpha \tilde{b}_\alpha \tilde{c}_\alpha$ for $1 \leq \alpha \leq r$. Let $A' = \langle \tilde{a}_\alpha \rangle$, write $a' = \dim A'$, and let $(a_1, ..., a_{a'}) = (a_i)$ be a basis of A'. Write $\tilde{a}_\alpha = \alpha^i_\alpha a_i$. Do the same for B', C'. We have the equations

$$\alpha^i_r \beta^k_r \gamma^l_r = \xi^\alpha \alpha^i_\alpha \beta^k_\alpha \gamma^l_\alpha \quad \forall i, k, l.$$

Now assume further that the t term vanishes, i.e., $x'_r = \xi^j x'_j$. Write

$$\begin{aligned} x'_\alpha &= a'_\alpha \tilde{b}_\alpha \tilde{c}_\alpha + \tilde{a}_\alpha b'_\alpha \tilde{c}_\alpha + \tilde{a}_\alpha \tilde{b}_\alpha c'_\alpha \\ &= \beta^k_\alpha \gamma^l_\alpha a'_\alpha b_k c_l + \alpha^i_\alpha \gamma^l_\alpha a_i b'_k c_l + \alpha^i_\alpha \beta^k_\alpha a_i b'_k c_l. \end{aligned}$$

We obtain the following equations relating a'_j, a'_r:

$$\beta^k_r \gamma^l_r a'_r \equiv \xi^j \beta^i_j \gamma^l_j a'_j \bmod A' \tag{3}$$

which span at most an $N_A := \dim \langle \tilde{b}_1 \otimes \tilde{c}_1, ..., \tilde{b}_r \otimes \tilde{c}_r \rangle$ dimensional space of equations (in particular, at most $b'c'$). The t-term vanishes iff all these equations and their B, C counterparts hold.

If the t term does not vanish we get (up to) N_A new elements of A appearing beyond the elements of A', and similarly for B, C.

If the t term vanishes, then there will be three types of vectors from A appearing in the t^2 term: elements of A', the a'_α that live in the solution space to (3) and appear as coefficients of II, and the a''_α which may span an N_A dimensional subspace.

5.1. Some Lemmas.

We assume $a' \leq b' \leq c'$ and use notations as above.

Lemma 5.1. *Assume dim $\langle a_1 b_1 c_1, ..., a_r b_r c_r \rangle < r$. then*

i. If $a' = 1$, then $N_A \leq r$.
ii. If $r = 4$, then $a' = 1$.
iii. If $r = 5$, then either $a' = b' = 1$ or $a' = b' = c' = 2$.
iv. If $r = 6$ then either $a' = 1$ or $c' \leq 3$.

Proof. All cases are similar, we write out ii. to give a sample calculation. We assume $a' = b' = c' = 2$ and derive a contradiction. For the first 3 points there are two possible normal forms: $a_1b_1c_1, a_2b_2c_2, (\alpha_1 a_1 + \alpha_2 a_2)(\beta_1 b_1 + \beta_2 b_2)(\gamma_1 c_1 + \gamma_2 c_2)$ and $a_1b_1c_1, a_1b_2c_2, a_2(\beta_1 b_1 + \beta_2 b_2)(\gamma_1 c_1 + \gamma_2 c_2)$, where a_1, a_2 are linearly independent elements of A, α_1, α_2 are constants and similarly for B, C. Write an unknown fourth point in the span of the first three in one of the normal forms as $(\alpha'_1 a_1 + \alpha'_2 a_2)(\beta'_1 b_1 + \beta'_2 b_2)(\gamma'_1 c_1 + \gamma'_2 c_2)$, expand out in monomials and see that there are no nontrivial solutions for the fourth point. □

Let

$$\begin{aligned} Sub_{a',b',c'} &:= \mathbb{P}\{T \in A \otimes B \otimes C \mid \\ &\exists A' \subseteq A, B' \subseteq B, C' \subseteq C, \dim A' = a',\ \dim B' = b',\ \dim C' = c',\ T \in A' \otimes B' \otimes C'\} \end{aligned}$$

and let $Sub^0_{a',b',c'}$ be the Zariski open subset of $Sub_{a',b',c'}$ consisting of elements not in any $Sub_{a'',b'',c''}$ with $a'' \leq a', b'' \leq b', c'' \leq c'$ and at least one inequality strict. Recall that $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subset Sub_{r,r,r}$.

Lemma 5.2. *Notations as above. If $x_1,...,x_r$ are limit points with $x_r=\xi^j x_j$ such that there exists curves $x_j(t)$ and $T\in Sub^0_{r,r,r}\subset A\otimes B\otimes C$ such that T appears in the limiting r-plane which corresponds to the t coefficient of the Taylor expansion, then for any curves with these limit points such that the t coefficient is zero, the contribution of II to the t^2 coefficient must be such that $II\equiv 0$ mod $A'\otimes B'\otimes C'$ for some A',B',C', each of dimension r and $x_j\in A'\otimes B'\otimes C'$.*

More generally, under the same circumstances, if there is T with $T\in Sub^0_{r-j,rr}$, Then $II\equiv 0$ mod $A''\otimes B'\otimes C'$ where $A'\subset A''$ and $dim(A''/A')\leq j$.

In particular, if $N_A+a'\geq r$, then $II\equiv 0$ mod $A'\otimes B\otimes C$ and similarly for permuations.

Similar results hold for higher order invariants.

Proof. We have r curves in each of the vector spaces A,B,C and are examining their derivatives. The $a''\otimes b\otimes c+a\otimes b''\otimes c+a\otimes b\otimes c''$ type terms appearing in the x_j'' are of the same nature as the terms appearing in x_j', so they can define a point of $Sub^0_{r,r,r}$. Thus, were $II\neq 0$ mod $A'\otimes B'\otimes C'$, we could alter our Taylor series to get a point of σ_r not lying in Sub_{rrr}, a contradiction. The other assertions are similar. □

Lemma 5.3. *Let $L^s\subset A\otimes B$ be spanned by $a_1b_1,...,a_sb_s$ and assume all $b_1,...,b_s$ are linearly independent. Then there is a linear combination of $a_1b_1,...,a_sb_s$ with all coefficients nonzero that is in $Seg(\mathbb{P}A\times\mathbb{P}B)$ iff $dim\langle a_1,...,a_s\rangle=1$.*

The proof is straightforward. The following lemmas are consequences of the above lemmas:

Lemma 5.4. *If $a'=b'=1$ and $r-1=dim\,C'$ then $II\equiv 0$ mod $A'\otimes B'\otimes C'$.*

Lemma 5.5. *If $a'=1$ and there is only one relation among $x_1,...,x_r$, then the terms arising from II mod $A'\otimes B'\otimes C'$ must be in $A'\otimes B\otimes C$. Moreover, in this situation, we must have $III\equiv 0$ mod $A'\otimes B'\otimes C'+Image\,II$.*

If there are p relations among the x_s then at most $p-1$ vectors which are independent as elements of A/A' can appear in II and III.

5.2. **Matrix multiplication.** Throughout this section we assume without loss of generality that $a'\leq b'\leq c'$.

Write the matrix multiplication operator for two by two matrices as $M=\phi_1+\phi_2$, and assume $\phi_1\in\sigma_r$, $\phi_2\in\sigma_{6-r}$ and ϕ_1 is not in σ_r^0 or any of the cases treated in [1]. We rule out the remaining cases. We must examine the cases $r=3,...,6$.

5.3. **Case $r=3$.** Any trisecant line to the Segre must be in it. Hence we may assume $(a',b',c')=(1,1,2)$. Assume the first nonzero term is the t coefficient. We have

$$p=p_1a_1b_1c_1+p_2a_1b_1c_2+p_3[a_{11}b_1c_1+a_1b_{11}c_1+a_1b_1c_{11}+a_{12}b_1c_2+a_1b_{12}c_2]$$

If any of the triples a_1,a_{11},a_{12}, b_1,b_{11},b_{12}, c_1,c_2,c_{11} fail to be linearly independent, we use the $\mathfrak{S}_3$ action to make that space the target of M (say it is C, i.e., $M:A^*\times B^*\to C$). Take $b\in\mathrm{Rker}\phi_2$ and $a\in\mathrm{Lker}\phi_2$ and consider the maps $M(\cdot,b)=\phi_1(\cdot,b)$ and $M(a,\cdot)=\phi_1(a,\cdot)$. Both $M(A^*,b),M(a,B^*)$ are nontrivial ideals, but if c_1,c_2,c_{11} span a two dimensional space the ideals coincide and we have a contradiction. So all three of the triples are linearly independent. Now examine the matrix of the map $M(\cdot,b)$ with respect to the bases a_1,a_{11},a_{12} and c_1,c_2,c_{11}:

$$\begin{pmatrix} p_1b_1(b)+p_3b_{11}(b) & p_3b_1(b) & 0\\ p_2b_1(b)+p_3b_{12}(b) & 0 & p_3b_1(b)\\ p_3b_1(b) & 0 & 0\end{pmatrix}$$

This matrix must have rank two, but that is impossible, a contradiction. By lemma 5.4, nothing changes if the t coefficient is zero.

5.4. **Case $r=4$.** By lemma 5.1 we must consider the cases $(1,1,2),(1,1,3)$ and $(1,2,2)$.

5.4.1. *Subcase* $(a',b',c')=(1,1,x)$. The case $x=2$ is the same as above except take $b\in \mathrm{Rker}\phi_2\cap B'^{\perp}$, $c\in \mathrm{Lker}\phi_2\cap C'^{\perp}$. For the $x=3$ case assume for the moment that the t coefficient is nonzero, we have

$$p\in\langle a_1b_1c_1, a_1b_1c_2, a_1b_1c_3, [a_{11}b_1c_1+a_1b_{11}c_1+a_1b_1c_{11}+a_{12}b_1c_2+a_1b_{12}c_2+a_{13}b_1c_3+a_1b_{13}c_3]\rangle.$$

Consider $M: B^*\times C^*\to A$ Take $b\in \mathrm{Lker}\phi_2\cap {b_1}^{\perp}$. Then $M(b,C^*)=\langle a_1\rangle$ a one dimensional ideal, a contradiction. Lemma 5.4 shows the same argument holds if the t coefficient is zero.

5.4.2. *Subcase* $(a',b',c')=(1,2,2)$. In this case we have a hyperplane in $\mathbb{C}^2\otimes\mathbb{C}^2$, thus it intersects the smooth quadric hypersurface $\mathbb{P}^1\times\mathbb{P}^1$ in a plane conic. Up to equivalence, there are two cases, depending on whether the conic is smooth or the union of two lines. The second case reduces to (a sum of cases with) $b'=1$. Assuming that no three points are colinear, we have the normal form $x_1=a_1b_1c_1$, $x_2=a_1b_2c_2$, $x_3=a_1(b_1+b_2)(c_1+c_2)$, $x_4=a_1(\beta^1b_1+\beta^2b_2)(\gamma^1c_1+\gamma^2c_2)$ with $\beta^1\gamma^2=\beta^2\gamma^1$. The t term is $x_1\wedge x_2\wedge x_3$ wedged against

$$\begin{aligned}&a_4'(\beta^1b_1+\beta^2b_2)(\gamma^1c_1+\gamma^2c_2)+a_1b_4'(\gamma^1c_1+\gamma^2c_2)+a_1(\beta^1b_1+\beta^2b_2)c_4'\\&-\beta^1\gamma^2[a_3'(b_1+b_2)(c_1+c_2)+a_1b_3'(c_1+c_2)+a_1(b_1+b_2)c_3']\\&-\beta^1(\gamma^1-\gamma^2)[a_1'b_1c_1+a_1b_1'c_1+a_1b_2c_1']-\beta^2(\gamma^2-\gamma^1)[a_2'b_2c_2+a_2b_2'c_2+a_2b_2c_2']\\&=\tilde a_1'b_1c_1+\tilde a_2'b_2c_2+\tilde a_3'(b_1c_2+b_2c_1)+a_1\tilde b_1'c_1+a_1\tilde b_2'c_2+a_1b_1\tilde c_1'+a_1b_2\tilde c_2'\end{aligned}$$

where $\tilde a_1'=\beta^1\gamma^1a_4'-\beta^1\gamma^2a_3'-\beta^1(\gamma^1-\gamma^2)a_1'$ etc...

If we take a nonzero $b\in\mathrm{Lker}\phi_2$, the map $M(\cdot,b): A^*\to C$ must have rank four or two. We may require additionally that $\tilde b_2'(b)=0$, in which case $M(\cdot,b)$ has rank at most three and therefore two. But then the only way to obtain a map of rank two is for $\tilde b_1'(b)=0$, and we have a nontrivial left ideal $M(A^*,b)=C'$. But now consider $a\in\mathrm{Rker}(\phi_2,a_1)$. $M(a,B^*)=C'$ and we have a contradiction.

From this normal form one can check directly that $II\equiv 0 \bmod A'\otimes B'\otimes C'$ so there is no need to consider the case where the t coefficient is zero.

5.5. **Case** $r=5$**.** By lemma 5.1 we are reduced to $(1,1,x),(1,2,2),(2,2,2),(1,2,3),(1,3,3)$.

Remark 5.6. If $a',b',c'\le 2$, then M cannot correspond to using the t term in ϕ_1. To see this, we may take $a\in A^*, b\in B^*$ such that $M(a,B^*),M(A^*,b)\subseteq C'$ by having a annhilate ϕ_2 and A' and similarly for b. Since $\dim C'=2$ we must have $M(a,B^*)=M(A^*,b)=C'$, a contradiction.

5.5.1. *Subcase* $(a',b',c')=(1,1,x)$. Argue as in §5.4.1. (At most one new vector from each of A,B can appear at any given level over the $r=4$ case but we are allowed to annhilate one new vector here as ϕ_2 is smaller.)

5.5.2. *Subcase* $(a',b',c')=(1,2,2)$. Consider the t^2 term. Let $\tilde a'$ be the unique vector mod A' that appears from one derivative. Consider $M: B^*\times C^*\to A$. Taking $b\in B'^{\perp}\cap\mathrm{Lker}\phi_2$, we have $M(b,C^*)\subseteq\langle A',\tilde a'\rangle$, but taking $c\in C'^{\perp}\cap\mathrm{Rker}\phi_2$, we have $M(B^*,c)\subseteq\langle A',\tilde a'\rangle$, a contradiction. The t^3 case follows as $III\equiv 0 \bmod \langle A',\tilde a'\rangle\otimes B\otimes C$ so we may use the same argument.

5.5.3. *Subcase* $(a',b',c')=(2,2,2)$. N_A,N_B,N_C each is 3 or 4 and in any of these cases lemma 5.2 shows $II\equiv 0\bmod A'\otimes B'\otimes C'$.

5.5.4. *Subcases* $(a',b',c')=(1,2,3),(1,3,3)$. Let $\phi_2=a_6b_6c_6$. Then taking $b\in B'^{\perp}$, $c\in C'^{\perp}$ we have $M(b,C^*)=M(B^*,c)=\langle A',a_6\rangle$ by lemma 5.5.

5.6. **Case** $r=6$**.** The same argument as in remark 5.6 shows that if $a',b',c'\le 3$ with at least one of them at most two, then M cannot be obtained by using the t term in ϕ_1.

5.6.1. *Subcases* $(a', b', c') = (1, 2, 2), (1, 2, 3), (1, 3, 3)$. Same proof as 5.5.2 works, for the last two we are allowed additional vectors in B', C' as there is no ϕ_2.

5.6.2. *Subcase:* $(a', b', c') = (1, 1, x)$. Argue as in §5.4.1.

5.6.3. *Subcase:* $(a', b', c') = (2, 2, 2)$. Here if II is nonzero modulo $A' \otimes B' \otimes C'$ there is at most one new vector in each space appearing at the level of first derivatives by lemma 5.2. Call these vectors $\tilde{a}', \tilde{b}', \tilde{c}'$. Then take $b \in \langle B', \tilde{b}' \rangle^{\perp}$, $c \in \langle C', \tilde{c}' \rangle^{\perp}$, we have $M(b, C^*) = M(B^*, c) = A'$. III cannot be nonzero modulo $A' \otimes B' \otimes C'$ by lemma 5.2.

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